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A METHOD OF CONSTRUCTING POLHODES OF AN INTERMEDIATE MOTION IN THE DYNAMICS OF A RIGID BODY*

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Asymptotic methods are used to construct the polhodes of an intermediate motion of a non-symmetric body about its centre of mass. The fundamental effects of this motion are governed by the action of the small external resistance of the medium, linear with respect to the angular velocity of rotation. Non-Eulerian motion is employed to construct the equations in osculating variables. A modification of the averaging procedure is proposed which makes it possible to obtain finite expressions for the polhodes of the intermediate motion. Results of the analysis of the intermediate motion and of the evolution of the polhodes of an Eulerian rotation of the body are given.

1. We consider the problem of the rapid motion of a non-symmetric rigid body about its centre of mass, whose basic effects are governed by the action due to the resistance of the surrounding medium, which is linear with respect to the angular velocity. Following /1/, we shall call the motions rapid, if the moment of external forces about a fixed point is small compared with the current value of the kinetic energy of rotation. We shall write the dynamic Euler equations, taking into account the specific features of the motion described earlier, in the form

$$Ap^{\cdot} + (C - B)\,qr = \varepsilon M_1\,(pqr,\ ABC,\ 123) \tag{1.1}$$

Here p, q, r are the projections of the angular velocity vector $\boldsymbol{\omega}$ onto the coordinate axes, A, B, C are the principal central moments of inertia of the body, ε is a small non-negative parameter, and $M_i\ (i = 1, 2, 3)$ are the components of the perturbing moment $\mathbf{M}$ where $\mathbf{M} = -I\boldsymbol{\omega}$, I is the matrix of the constant coefficients /2/ of resistance I_{ij} in associated axes $(i,\ j = 1, 2, 3)$. Henceforth we shall assume that $A > B > C$.

When studying the evolution of rapid motions of a rigid body about the centre of mass, we normally use the Euler-Poinsot motion as the generating motion obtained from Eqs.(1.1) for $\varepsilon = 0$, and we apply the method of varying the arbitrary Lagrange constants /1-5/ (of the generating solution). At the same time, the universal character of the Lagrange's method /5/ which can be used when choosing the unperturbed motion arbitrarily, makes it possible to carry out the investigation using motions resembling that described by Eqs.(1.1) more closely than the Eulerian motion. Such motions, which were first encountered in classical celestial mechanics, have become particularly valuable in connection with constructing the theory of the motion of artificial celestial bodies, and are called intermediate motions, while the corresponding trajectories are called intermediate orbits /3, 5, 5/.

The problem of constructing the trajectories (polhodes and herpolhodes) of the intermediate motion of a rigid body was discussed in /3/. The method involves taking into account the most significant special features of the rotational motion in such a manner that the corresponding equations can be integrated in closed form. The present paper gives a method of constructing the polhodes of the intermediate motion, taking into account the small forces opposing the rotation of the body.

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2. The process of integrating the dynamic Euler equations reduces, in the Euler-Poinsot case, to finding the general solution of the equations

$$q^{\cdot 2} = F_4(q),\quad F_4(q) = \sigma^2(\lambda_1^2 - q^2)(\lambda_2^2 - q^2) \tag{2.1}$$
$$\sigma^2 = (A-B)(B-C)/(AC)$$

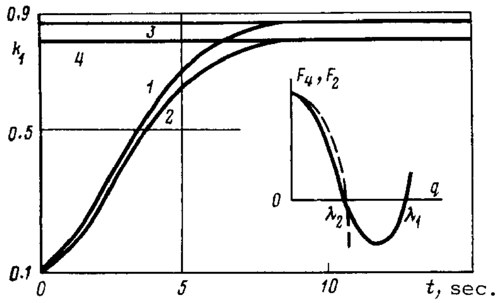

Fig.1

written using the notation of /7/ where $F_4(q)$ is a fourth-degree polynomial in q.

Let us consider the case $|\lambda_1| > |\lambda_2|$ corresponding to ω appearing near the axis of the moment of inertia C. We approximate the polynomial $F_4(q)$ from the right-hand side of (2.1) over the interval $[-\lambda_2, \lambda_2]$, using the second degree polynomial $F_2(q)$. The insert in Fig.1 shows one of the branches of the corresponding even functions $F_4 = F_4(q)$ (the solid line) and $F_2 = F_2(q)$ (the dashed line). Then retaining the previous notation for the projections of ω, we shall write

$$p = s_1\lambda_2\sqrt{\frac{B(B-C)}{A(A-C)}}\cos u_1,\quad q = s_2\lambda_2\sin u_1 \tag{2.2}$$
$$r = s_3\lambda_1\sqrt{\frac{B(A-B)}{C(A-C)}}\sqrt{1-k_1^2\sin^2 u_1},\quad k_1^2 = \frac{\lambda_2^2}{\lambda_1^2}$$
$$u_1 = \lambda_1\sigma t + \alpha$$

where α is an arbitrary integration constant and $s_i\ (i = 1, 2, 3)$ are numbers equal to ± 1 and determining the signs of the radicals /8/.

We note that expressions of the form (2.2) can be obtained from the corresponding Euler-Poinsot equations by retaining the first harmonic in the Fourier series /9/, with an accuracy of $O(k_1^2)$, or by using the Landen transformation /6/.

When solving the fundamental problem of constructing the equations of motion /10/ using the known first integrals (2.2), we shall write

$$Ap^{\cdot} + (C-B)\,qr\delta = 0 \quad (pqr,\ ABC) \tag{2.3}$$
$$\delta = \left[1 + \frac{B(A-B)q^2}{C(A-C)r^2}\right]^{1/2}$$

Eqs.(2.3) and their general solution (2.2), which will be used below to study the behaviour of soltuions of perturbed system (1.1), will be called, unlike the generating equations, the basis equations.

In order to study the evolution of motion, we construct the equations in osculating elements /5/ using arbitrary constants of the basis solution(2.2) as the latter. After carrying out the necessary manipulations, we write the equations in the following form:

$$\lambda_1^{\cdot} = \varepsilon\left[k_1\frac{M_2}{B}\sin u_1 - \sqrt{\frac{C(A-C)}{B(A-B)}}\frac{M_3}{C}\sqrt{1-k_1^2\sin^2 u_1}\right] \tag{2.4}$$
$$\lambda_2^{\cdot} = \varepsilon\left[\sqrt{\frac{A(A-C)}{B(B-C)}}\frac{M_1}{A}\cos u_1 + \frac{M_2}{B}\sin u_1\right]$$
$$u_1^{\cdot} = \lambda_1\sigma\sqrt{1-k_1^2\sin^2 u_1} + \frac{\varepsilon}{\lambda_2}\left[\frac{M_2}{B}\cos u_1 - \sqrt{\frac{A(A-C)}{B(B-C)}}\frac{M_1}{A}\sin u_1\right]$$

where we have assumed that $s_1 = s_2 = 1,\ s_3 = -1$.

Let us investigate the solution of system (2.4) for small ε over a long time interval $t \sim \varepsilon^{-1}$, using the method of averaging /11, 12/. We note that Eqs.(2.4) refer to general-type systems whose averaging is carried out /11/ along the trajectories of the perturbed Euler-Poinsot motion, which can easily be confirmed for (2.4) was used in /13/ to reduce the averaging manipulations to determining the quadratures of elementary functions. In order to simplify the averaging procedure we shall introduce the average motion /5/ $n_1 = 2\pi/T_1$ and angular velocity vector where $T_1 = 4K/(\lambda_1\sigma)$ is the period of a single circuit of the polhodes in the unperturbed motion. As a result we can write, retaining the former notation for the variables, the approximate system with rotating phase (we shall call it system A) in the form (2.4), where the first term in the last equation is replaced by n_1.

The passage to the average motion in the last equation corresponds to approximating the generating Euler-Poinsot solution, in the process of averaging (2.4), by its approximate expression (2.2) where $u_1^{\cdot} = n_1$. We also note the overlap of the trajectory ω, with the periods of generating motions for Eqs.(2.4) and system A.

In connection with the proposed change in the averaging procedure, we should take notice of /14/ where the investigation of non-linear systems of differential equation in standard form using the method of averaging was simplified by replacing the initial equations by approximate equations constructed using the first term of the expansion of the generating solution in series in powers of some parameter. Expansion in a Fourier series was recommended for the case of periodic generating solutions.

3. We propose that system A be used to construct the polhodes of the intermediate motion. Let us write the averaged system of the first approximation by substituting into the right-hand sides of system A the expression for the projections of the perturbing moment taking (2.2) into account, and average over the rapid variable u_1. The error in the averaged solution for the slow variable of the system A is of the order of ε in the time interval over which ω will execute $\sim \varepsilon^{-1}$ rotations over the polhodes.

We shall write the averaged equations for the slow variables in the form

$$\lambda_1^{\cdot} = -\frac{\varepsilon}{2\lambda_1}\left[\frac{I_{22}}{B}\lambda_2^2 + \frac{I_{33}}{C}(2\lambda_1^2 - \lambda_2^2)\right] \qquad (3.1)$$

$$\lambda_2^{\cdot} = -\frac{\varepsilon\lambda_2}{2}\left(\frac{I_{11}}{A} + \frac{I_{22}}{B}\right)$$

In using the proposed approach to the choice of the basis solution and modifying the averaging procedure, we find that it is possible to obtain the solutions for (3.1) in the finite form

$$\lambda_1^2(t) = \{C_1 + C_2\alpha_1[\exp(\alpha_2\varepsilon t) - 1]\}\exp\left(-2\frac{I_{33}}{C}\varepsilon t\right) \qquad (3.2)$$

$$\lambda_2(t) = C_2\exp\left[-\frac{1}{2}\left(\frac{I_{11}}{A} + \frac{I_{22}}{B}\right)\varepsilon t\right]$$

$$\alpha_1 = \left(\frac{I_{33}}{C} - \frac{I_{22}}{B}\right)\left(2\frac{I_{33}}{C} - \frac{I_{11}}{A} - \frac{I_{22}}{B}\right)^{-1}, \quad \alpha_2 = \left(2\frac{I_{33}}{C} - \frac{I_{11}}{A} - \frac{I_{22}}{B}\right)$$

Here $C_i\ (i = 1, 2)$ are arbitrary constants of integration which can be found from the formulas $C_1 = \lambda_1^2(0)$, $C_2 = \lambda_2(0)$.

When $|\lambda_1| < |\lambda_2|$, we must interchange in (2.4) and below λ_1 with λ_2, I_{11}/A and I_{33}/C, k_1 with k_2 respectively, where $k_2 = k^{-1}_1$.

Expressions (2.2) where the values of $\lambda_1(t), \lambda_2(t)$ and $k_1(t)$ vary slowly in accordance with relations (3.2), define the polhodes of the intermediate motion of the body about the centre of mass. The intermediate motion is determined uniquely by choosing the law of motion of the angular velocity vector over the polhodes.

4. Let us compare the polhodes of the intermediate motion determined by the expressions (2.2), (3.2), with the results of /1/. The study of the intermediate motion can be reduced to analysing the equation in k_1

$$k_1^{\cdot 2} = \varepsilon k_1^2\left[\left(2\frac{I_{33}}{C} - \frac{I_{11}}{A} - \frac{I_{22}}{B}\right) - k_1^2\left(\frac{I_{33}}{C} - \frac{I_{22}}{B}\right)\right] \qquad (4.1)$$

The corresponding equation /1/ in k_1 describing the evolution of the polhodes of the Eulerian rotation of the body, is written in the form

$$k_1^{\cdot 2} = 2\varepsilon\left[k_1^2\left(\frac{I_{33}}{C}\frac{\mathbf{E}}{\mathbf{K}} + \frac{I_{22}}{B}\frac{\mathbf{K} - \mathbf{E}}{\mathbf{K}} - \frac{I_{11}}{A}\right) + \frac{\mathbf{K} - \mathbf{E}}{\mathbf{K}}\left(\frac{I_{11}}{A} - \frac{I_{22}}{B}\right)\right] \qquad (4.2)$$

where $\mathbf{K} = \mathbf{K}(k_1)$, $\mathbf{E} = \mathbf{E}(k_1)$ are total elliptic integrals of first and second kind respectively, and k_1 is the modulus of the elliptic Jacobi functions. It can be shown that when we take into account the additional condition for the existence of the quasistationary non-zero solutions of (4.1)

$$\left(2\frac{I_{33}}{C} - \frac{I_{11}}{A} - \frac{I_{22}}{B}\right)\left(\frac{I_{33}}{C} - \frac{I_{22}}{B}\right)^{-1} \leqslant 1$$

then the results of a qualitative analysis of the behaviour of the solutions of (4.1) are identical with those given in /1/ for Eq.(4.2).

The graphs in Fig.1 showing the solution of Eqs.(4.2) by curve *1* and of (4.1) by curve *2* for $I_{11}/A = 3.5$, $I_{22}/B = 1.5$, $I_{33}/C = 3.0$, $k_1(0) = 0.1$, provide a numerical comparison. Curves *3* and *4* correspond to the quasistationary solutions. Fig.2 shows the relations connecting the quasi-stationary solutions with the values of the dissipation coefficients, where curves *1* correspond to Eq.(4.2) and curve *2* to (4.1). The solid lines represent the dependence on I_{11}/A, where we have assumed that $I_{22}/B = 1.5$, $I_{33}/C = 3.0$, the dashed lines the dependence on I_{22}/B for $I_{11}/A = 2.5$, $I_{33}/B = 2.0$, and the dot-dash lines the dependence on I_{33}/C for $I_{11}/A = 4.0$, $I_{22}/B = 0.2$.

The above results show that the difference between the polhodes of the intermediate motion constructed here and the corresponding evolutionary changes in the polhodes of the Eulerian rotation /1/ does not exceed 10% in these cases. The error arises when averaging the right-hand sides of system A along the polhodes (2.2), by using the law of motion of ω along it of the form $u_1^{\cdot} = n_1$ instead of the expression $u_1^{\cdot} = \lambda_1 \sigma \sqrt{1 - k_1^2 \sin^2 u_1}$ corresponding to the Euler-Poinsot motion.

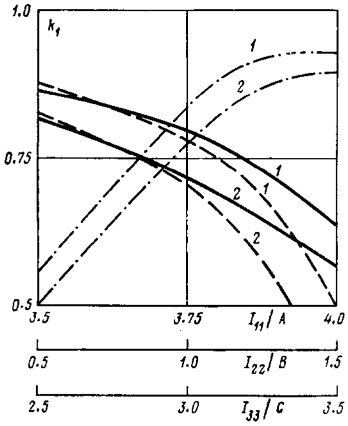

Fig.2

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